Essential Spectrum of Schrödinger Operators with no Periodic Potentials on Periodic Metric Graphs

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The main aim of the talk is the investigation of the essential spectrum of the quantum graphs. For this aim we use *the limit operators method (see for instance the book)*

- *V.S.Rabinovich, S. Roch, B.Silbermann, Limit Operators and its Applications in the Operator Theory, In ser. Operator Theory: Advances and Applications, vol 150, ISBN 3-7643-7081-5, Birkhäuser Velag, 2004, 392 pp.*

Earlier this method was successfully applied to the study of the essential spectrum of electromagnetic Schrödinger and Dirac operators on $\mathbb{R}^n$ for wide classes of potentials. In particular, a very simple and transparent proof of the Hunziker-van Winter-Zhislin Theorem (HWZ-Theorem) for multi-particle Hamiltonians has been obtained.

- *V. Rabinovich, Essential spectrum of perturbed pseudodifferential operators. Applications to the Schrödinger, Klein-Gordon, and Dirac operators, Russian Journal of Math. Physics, Vol.12, No.1, 2005, p. 62-80*

The limit operators method also was applied to the study of the location of the essential spectrum of discrete Schrödinger and Dirac operators on $\mathbb{Z}^n$, and on periodic combinatorial graphs.

- *V.S. Rabinovich, S. Roch, The essential spectrum of Schrödinger operators on lattice, Journal of Physics A, Math. Theor. 39 (2006) 8377-8394*
- *V.S. Rabinovich, S. Roch, Essential spectra of difference operators on Z^n-periodic graphs, J. of Physics A: Math. Theor. ISSN 1751-8113, 40 (2007) 10109–10128*

Periodic metric graphs

We consider a periodic metric graph Γ embedded in $\mathbb{R}^n$. We suppose that a graph Γ consists of a countably infinite set of vertices $\mathcal{V} = \{v_i\}_{i \in \mathcal{I}}$ and a set $\mathcal{E} = \{e_j\}_{j \in J}$ of edges connecting these vertices. Each edge e is a line segment

$$[\alpha, \beta] = \{x \in \mathbb{R}^2 : x = (1 - \theta)\alpha + \theta\beta, \theta \in [0, 1]\} \subset \mathbb{R}^2$$

connecting its endpoints (vertices α, β), and we suppose that for the every pair of vertices $\{\alpha, \beta\}$ there exists not more than one edge connecting this pair. Let $\mathcal{E}_v$ be a set of edges incident to the vertex v (i.e., containing v). We will always assume that the degree (valence) $d(\nu)$ (the number of points of $\mathcal{E}_v$) of any vertex v is finite and positive. Vertices with no incident edges are not allowed.

For each edge $e = [\alpha, \beta]$ we assign its length $l_e = \|\alpha - \beta\|_{\mathbb{R}^n} < \infty$. We also suppose that the graph Γ is a connected set. The graph is a metric space with a metric induced by the standard metric of $\mathbb{R}^n$. The topology on Γ is induced also by the topology on $\mathbb{R}^n$, and the measure dl on Γ is the line Lebesgue measure on every edge.

We suppose that on the graph $\Gamma \subset \mathbb{R}^n$ acts a group $\mathbb{G}$ isomorphic to $\mathbb{Z}^m, 1 \leq m \leq n$, that is

$$\mathbb{G} = \left\{ g \in \mathbb{R}^n : g = \sum_{j=1}^{m} \alpha_j \mathfrak{e}_j, \alpha_j \in \mathbb{Z}, \mathfrak{e}_j \in \mathbb{R}^n \right\}$$

where the system $\{\mathfrak{e}_1, ..., \mathfrak{e}_m\}$ is linear independent. The group $\mathbb{G}$ acts on Γ by the shifts

$$\mathbb{G} \times \Gamma \ni (g, x) \to g + x \in \Gamma,$$

where $g + x$ is the sum of the vectors in $\mathbb{R}^n$. We suppose that the group $\mathbb{G}$ acts *freely* on X, that is if $g + x = x$ for some $x \in \Gamma$, then $g = 0$. Moreover we suppose that the action of $\mathbb{G}$ on Γ is co-compact, that is the fundamental domain $\Gamma_0 = \Gamma/\mathbb{G}$ of Γ with respect to the action of $\mathbb{G}$ on Γ is a compact set in the corresponding quotient topology. Let $G_0 \subset \Gamma$ be a measurable set with the compact closure which contains for every $x \in \Gamma$ exactly one element of the quotient class $x + \mathbb{G} \in \Gamma/\mathbb{G}$. There exists a natural one-to-one mapping $G_0 \to \Gamma/\mathbb{G}$ which is the composition of the inclusion mapping $G_0 \subset \Gamma$ and the canonical projection $\Gamma \to \Gamma/\mathbb{G}$.

Let $G_h = G_0 + h$, $h \in \mathbb{G}$. Then

$$G_{h_1} \cap G_{h_2} = \varnothing \quad \text{if } h_1 \neq h_2,$$

and

$$\bigcup_{h \in \mathbb{G}} G_h = \Gamma.$$

We say that the graph Γ is *periodic with respect to* $\mathbb{G}$ if the above given conditions are satisfied.

We denote by $L^2(\Gamma)$ the space of measurable functions on Γ with the norm

$$\|u\|_{L^2(\Gamma)} = \left(\int_\Gamma |u(x)|^2 \, dx \right)^{1/2} = \left(\sum_{e \in \mathcal{E}} \int_e |u(x)|^2 \, dx \right)^{1/2}$$

and the scalar product

$$\langle u, v \rangle = \sum_{e \in \mathcal{E}} \int_e u(x) \bar{v}(x) dx.$$

Schrödinger operators on on the periodic graph

Let $\Gamma \subset \mathbb{R}^n$ be a periodic with respect to $\mathbb{G}$ metric graph. We denote by $H^s(e)$, $e \in \mathcal{E}$, $s \in \mathbb{R}$ the Sobolev space on the edge e, and let

$$H^s(\Gamma) = \bigoplus_{e \in \mathcal{E}} H^s(e)$$

with the norm

$$\|u\|_{H^s(\Gamma)} = \left(\sum_{e \in \mathcal{E}} \|u_e\|^2_{H^s(e)} \right)^{1/2}.$$

We denote $\mathcal{E}_v$ the set of edges incident v, and let $d(\nu) \in \mathbb{N}$ be a number of the edges in $\mathcal{E}_v$ (The periodicity of the graph Γ implies that $d(\nu + g) = d(\nu)$ for every $\nu \in \mathcal{V}$ and $g \in \mathbb{G}$).

We consider the Schrödinger operator on Γ

$$Hu(x) = -\frac{d^2u(x)}{dx^2} + q(x)u(x), x \in \Gamma \backslash \mathcal{V}, \quad (1)$$

where $q \in L^\infty(\Gamma)$. We provide the operator H by the Kirchhoff-Neumann conditions at the every vertex $v \in \mathcal{V}$.

$$u_e(v) = u_{e'}(v), \text{ if } e, e' \in \mathcal{E}_v, \text{ and} \sum_{e \in \mathcal{E}_v} u'_e = 0 \quad (2)$$

where the orientations of the edges $e \in \mathcal{E}_v$ are taken as outgoing from v.

By the usual way we obtain that

$$\operatorname{Re}\langle Hu, u\rangle \geq m_q \|u\|^2_{L^2(\Gamma)}, u \in \tilde{H}^2(\Gamma), m_q = \inf_{x\in\Gamma} \operatorname{Re} q(x). \qquad (3)$$

This property implies that the operator H provided by the Kirchhoff-Neumann conditions (2) defines an unbounded closed operator $\mathcal{H}$ in $L^2(\Gamma)$ with the domain $\tilde{H}^2(\Gamma)$, and $\mathcal{H}$ is a selfadjoint operator if the potential q is a real-valued function.

We recall that a closed unbounded operator A acting in the Hilbert space X with dense domain D_A is called a Fredholm operator if $\ker A$ is a finite dimensional sub-space of X, $\mathrm{Im}\, A$ is closed in X, and $X / \mathrm{Im}\, A$ is a finite-dimensional space. We introduce in $X_1 = D_A$ the norm of the graphics

$$\|u\|_{D_A} = \left(\|u\|_X^2 + \|Au\|_X^2 \right)^{1/2} . \tag{4}$$

Since A is closed, X_1 is a Banach space. Then A is a Fredholm operator as unbounded operator in X if and only if $A : X_1 \to X$ is a Fredholm operator as a bounded operator.

Note that the norm in $\tilde{H}^2(\Gamma)$ equivalents to the graphic norm in $D_{\mathcal{H}}$

$$\|u\|_{D_{\mathcal{H}}} = \left(\|u\|^2_{L^2(\Gamma)} + \|Hu\|^2_{L^2(\Gamma)}\right)^{1/2}$$

since the potential $q \in L^{\infty}(\Gamma)$. Hence the Fredholmness of the operator $\mathcal{H}$ as an unbounded operator in $L^2(\Gamma)$ with domain $\tilde{H}^2(\Gamma)$ is equivalent to the Fredholmness of $\mathcal{H}$ as a bounded operator from $\tilde{H}^2(\Gamma)$ into $L^2(\Gamma)$. We recall that the essential spectrum $sp_{ess}\mathcal{H}$ of $\mathcal{H}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathcal{H} - \lambda I$ is not Fredholm operator as unbounded in $L^2(\Gamma)$ with domain $\tilde{H}^2(\Gamma)$. Note that for a self-adjoint operator $\mathcal{H}$

$$sp_{dis}\mathcal{H} = sp\mathcal{H} \setminus sp_{ess}\mathcal{H}.$$

Limit operators

Let $h \in \mathbb{G}$. Then the shift (translation) operators

$$V_h u(x) = u(x - h), x \in \Gamma, h \in \mathbb{G}$$

are isometric operators in $L^2(\Gamma)$ and $H^2(\Gamma)$. Moreover if $u \in H^2(\Gamma)$ satisfies the Kirchhoff-Neumann conditions at the every vertex $v \in \mathcal{V}$ the function $V_h u$ also satisfies these conditions for every $v \in \mathcal{V}$. Hence V_h is an isometric operator in $\tilde{H}^2(\Gamma)$.

Let $\mathbb{G} \ni h_k \to \infty$. We consider the family of operators

$$V_{-h_k}\mathcal{H}V_{h_k} : \tilde{H}^2(\Gamma) \to L^2(\Gamma)$$

defined by the Schrödinger operators

$$V_{-h_k}HV_{h_k}u(x) = (-\frac{d^2u(x)}{dx^2} + q(x+h_k))u(x), x \in \Gamma\backslash\mathcal{V}.$$

We say that the potential $q \in L^\infty(\Gamma)$ is rich, if for every sequence $\mathbb{G} \ni h_k \to \infty$ there exists a subsequence $\mathbb{G} \ni g_k \to \infty$ and a limit function $q^g \in L^\infty(\Gamma)$ such that

$$\lim_{k\to\infty} \sup_{x\in K\subset\Gamma} |q(x+g_k) - q^g(x)| = 0 \tag{5}$$

for every compact set $K \subset \Gamma$.

Example

Let $q \in C_{b,u}(\Gamma)$ the space of bounded uniformly continuous functions on Γ. If $q \in C_{b,u}(\Gamma)$ the sequence $\{q(x+h_k), x \in \Gamma, h_k \in \mathbb{G}\}$ is uniformly bounded and equicontinuous. Then by Arzela-Ascoli Theorem there exists a subsequence $\{q(x+g_k), x \in \Gamma, g_k \in \mathbb{G}\}$ such that (5) holds.

Essential spectrum of Schrödinger operators on periodic graphs and limit operators

Let $q \in L^{\infty}(\Gamma)$ be a potential and a sequence $\mathbb{G} \ni g_k \to \infty$ is such

$$\lim_{k\to\infty} \sup_{x\in K\subset\Gamma} |q(x+g_k) - q^g(x)| = 0 \tag{6}$$

for every compact set $K \subset \Gamma$ and a function $q^g \in L^{\infty}(\Gamma)$. Then the unbounded in $L^2(\Gamma)$ operator $\mathcal{H}^g$ with domain $\tilde{H}^2(\Gamma)$ generated by the Schrödinger operator

$$H^g u(x) = -\frac{d^2u(x)}{dx^2} + q^g(x)u(x), x \in \Gamma \backslash \mathcal{V}$$

is called the limit operator of $\mathcal{H}$ defined by the sequence $\mathbb{G} \ni g_k \to \infty$. We denote by $Lim(\mathcal{H})$ the set of all limit operators of the the operator $\mathcal{H}$.

The main result of the talk is:

Theorem

Let Γ be a periodic with respect to the group $\mathbb{G}$ metric graph and $\mathcal{H}_q$ be a Schrödinger operator in $L^2(\Gamma)$ with domain $\tilde{H}^2(\Gamma)$ with a rich potential $q \in L^\infty(\Gamma)$. Then

$$sp_{ess}\mathcal{H}_q = \bigcup_{\mathcal{H}_q^g \in Lim(\mathcal{H}_q)} sp\mathcal{H}_q^g.$$

Periodic potentials

Let Γ be a graph periodic with respect to the action of the group $\mathbb{G}$

$$\mathbb{G} = \left\{ g \in \mathbb{R}^n : g = \sum_{j=1}^{m} \alpha_j \mathfrak{e}_j, \alpha_j \in \mathbb{Z}, \mathfrak{e}_j \in \mathbb{R}^n \right\},$$

provided by the Schrödinger operator

$$H_q u(x) = -\frac{d^2 u(x)}{dx^2} + q(x)u(x), x \in \Gamma \backslash \mathcal{V}, \tag{7}$$

with the potential $q \in L^\infty(\Gamma)$ periodic with respect to the action of the group $\mathbb{G}$

$$q(x+g) = q(x), x \in \Gamma, g \in \mathbb{G}.$$

Since $\mathcal{H}_q$ is invariant with respect to shifts all limit operators $\mathcal{H}_q^h$ coincide with $\mathcal{H}_q$. Hence by Theorem 2

$$sp_{ess}\mathcal{H}_q = sp\mathcal{H}_q,$$

and the periodic operator does not have the discrete spectrum.

Let the potential $q \in L^{\infty}(\Gamma)$ be a periodic with respect to $\mathbb{G}$ *real-valued function*. Then $\mathcal{H}_q$ with domain $\tilde{H}^2(\Gamma)$ is a self-adjoint operator in $L^2(\Gamma)$ with the spectrum which has a band structure

$$sp\mathcal{H}_q = sp_{ess}\mathcal{H}_q = \bigcup_{j=1}^{\infty} \left[\alpha_j, \beta_j\right].$$

Degenerated at infinity perturbations

Let

$$q = q_0 + q_1,$$

where $q_0 \in L^\infty(\Gamma)$ is a periodic real-valued function, and $q_1 \in L^\infty(\Gamma)$ is a real valued functions such that

$$\lim_{\Gamma \ni x \to \infty} q_1(x) = 0.$$

Then

$$\mathcal{H}_q^g = \mathcal{H}_{q_0}$$

and hence

$$sp_{ess}\mathcal{H}_q^g = sp\mathcal{H}_{q_0}.$$

Hence only the discrete spectrum can be arise in the gaps of the spectrum of the periodic operator $\mathcal{H}_{q_0}$ under such sort impurities (pertrubations).

Slowly oscillating perturbations

We say that a function $a \in C_b(\Gamma)$ is slowly oscillating at infinity and belongs to the class $SO(\Gamma)$ if for every sequence $\mathbb{G} \ni g^m \to \infty$

$$\lim_{m\to\infty} \sup_{\{x_1,x_2\in\Gamma:|x_1-x_2|\leq 1\}} |a(x_1+g_m)-a(x_2+g_m)| = 0. \tag{8}$$

One can prove that $SO(\Gamma) \subset C_{b,u}(\Gamma)$.

Example

Let $f \in C_b^1(\mathbb{R})$, $a(x) = f((1+|x|)^\alpha), 0 < \alpha < 1, x \in \mathbb{R}^n$. Then $a\,|_\Gamma \in SO(\Gamma)$.

Let $a \in SO(\Gamma)$. Then every sequence $\mathbb{G} \ni h_m \to \infty$ has a subsequence $g_m \in \mathbb{G}$ such that for every $x \in \Gamma$ there exists a limit

$$a^g = \lim_m a(x + g_m),$$

and a^g independent of x.

We consider potentials of the form

$$q = q_0 + q_1,$$

where $q_0 \in L^\infty(\Gamma)$ is a periodic real-valued function, and q_1 is a real-valued function of the class $SO(\Gamma)$. Then the potential q is rich, and all limit operators are of the form

$$\mathcal{H}_q^g = \mathcal{H}_{q_0 + q_1^g}$$

where $q_1^g = \lim_{m\to\infty} q(x + g_m)$ and $q_1^g \in \mathbb{R}$ are independent of $x \in \Gamma$.

Then

$$sp\mathcal{H}_q^g = \bigcup_{j=1}^{\infty} \left[\alpha_j + q_1^g, \beta_j + q_1^g\right].$$

Let

$$m_{q_1}^{\infty} = \liminf_{\mathbb{G}\ni g\to\infty} q_1(x+g), M_{q_1}^{\infty} = \limsup_{\mathbb{G}\ni g\to\infty} q_1(x+g), x \in \Gamma,$$

where m_{q_1}, M_{q_1} are independent of the choice of $x \in \Gamma$.

Let $m > 1$. Then the set of the partial limits of the function $\mathbb{G} \ni g \to q_1(x+g) \in \mathbb{R}$ is a segment $\left[m_{q_1}^{\infty}, M_{q_1}^{\infty}\right]$. Applying formula

$$sp_{ess}\mathcal{H}_q = \bigcup_{\mathcal{H}_q^g \in Lim(\mathcal{H}_q)} sp\mathcal{H}_q^g$$

we obtain that

$$sp_{ess}\mathcal{H}_q = \bigcup_{j=1}^{\infty} \left[\alpha_j + m_{q_1}^{\infty}, \beta_j + M_{q_1}^{\infty}\right].$$

In the case $n = 1$ the set of the partial limits has two components $\left[m_{q_1}^{\pm\infty}, M_{q_1}^{\pm\infty}\right]$ and we obtain that

$$sp_{ess}\mathcal{H}_q = \bigcup_{j=1}^{\infty} \left[\alpha_j + m_{q_1}^{+\infty}, \beta_j + M_{q_1}^{+\infty}\right] \cup \left[\alpha_j + m_{q_1}^{-\infty}, \beta_j + M_{q_1}^{-\infty}\right].$$

We consider the gaps in the essential spectrum of $\mathcal{H}_q$

$$(\beta_j + M_{q_1}^{\infty}, \alpha_{j+1} + m_{q_1}^{\infty}), j = 1, ..., ...$$

Let

$$osc_{\infty}(q_1) = M_{q_1}^{\infty} - m_{q_1}^{\infty} > \alpha_{j_0+1} - \beta_{j_0}. \tag{9}$$

Then the gap $(\beta_{j_0} + M_{q_1}^{\infty}, \alpha_{j_0+1} + m_{q_1}^{\infty})$ disappears. If condition (9) is satisfied for all $j \in \mathbb{N}$ all gaps in the essential spectrum of $\mathcal{H}_q$ are disappear and all bands of the $sp_{ess}\mathcal{H}_q$ are overlapping. Hence

$$sp_{ess}\mathcal{H}_q = [\alpha_1, +\infty),$$

and

$$sp_{dis}\mathcal{H}_q \subset (m_q, \alpha_1 + m_{q_1}^{\infty}).$$

Let φ be a function defined on $\mathbb{R}^n$. Then we denote by $\hat{\varphi}$ the restriction of φ on the graph Γ.

Definition

We say that $A \in \mathcal{B}(L^2(\Gamma))$ belongs to the class $\mathcal{A}(\Gamma)$ if for every function $\varphi \in C_{b,u}(\mathbb{R}^n)$

$$\lim_{t\to 0} \left\| [A, \widehat{\varphi_t} I] \right\|_{\mathcal{B}(L^2(\Gamma))} = \lim_{R\to 0} \left\| A\widehat{\varphi_t} I - \widehat{\varphi_t} A \right\|_{\mathcal{B}(L^2(\Gamma))} = 0. \qquad (10)$$

It is easy to prove that $\mathcal{A}(\Gamma)$ is a C^*-subalgebra of $\mathcal{B}(L^2(\Gamma))$.
Let $N \in \mathbb{N}$, $[-N, N]_{\mathbb{Z}} = \{\alpha \in \mathbb{Z} : |\alpha| \leq N\}$, and

$$\mathbb{G}_N = \left\{ g \in \mathbb{R}^m : g = \sum_{i=1}^{m} \alpha_i \mathfrak{e}_i, \alpha_i \in [-N, N]_{\mathbb{Z}} \right\}.$$

We set

$$\Gamma_N = \bigcup_{g \in \mathbb{G}_N} G_g$$

and let $\mathbb{P}_N \in \mathcal{B}(L^2(\Gamma))$ be the operator of the multiplication by the characteristic function of Γ_N, and $\mathbb{Q}_N = I - \mathbb{P}_N$.

Definition

Let $A \in \mathcal{B}(L^2(\Gamma))$ and $\mathbb{G} \ni h_k \to \infty$. An operator $A^h \in \mathcal{B}(L^2(\Gamma))$ is called *a limit operator* of A defined by the sequence $h_k \in \mathbb{G}$, if for every $N \in \mathbb{N}$

$$\begin{aligned} \lim_{k\to\infty} \left\| \left(V_{-h_k} A V_{h_k} - A^h \right) \mathbb{P}_N \right\|_{\mathcal{B}(L^2(\Gamma))} &= 0, \qquad (11) \\ \lim_{k\to\infty} \left\| \mathbb{P}_N \left(V_{-h_k} A V_{h_k} - A^h \right) \right\|_{\mathcal{B}(L^2(\Gamma))} &= 0. \end{aligned}$$

We say that the operator A is **rich** if every sequence $\mathbb{G} \ni h_k \to \infty$ has a subsequence $\mathbb{G} \ni g_k \to \infty$ defining a limit operator A^g. We denote by $Lim(A)$ the set of all limit operators of A.

Definition

An operator $A \in \mathcal{B}(L^2(\Gamma))$ is called locally invertible at infinity if there exist $R \in \mathbb{N}$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L^2(\Gamma))$ such that

$$\mathcal{L}_R A \mathbb{Q}_R = \mathbb{Q}_R, \mathbb{Q}_R A \mathcal{R}_R = \mathbb{Q}_R.$$

Theorem

Let $A \in \mathcal{A}(\Gamma)$ and be rich. Then A is locally invertible at infinity if and only if all limit operators $A^h \in Lim(A)$ are invertible in $L^2(\Gamma)$.

Definition

We say that $A \in \mathcal{B}(L^2(\Gamma))$ is a locally Fredholm operator if for every $R \in \mathbb{N}$ there exits operators $\mathcal{L}_R, \mathcal{R}_R$ such that

$$\mathcal{L}_R A \mathbb{P}_R = \mathbb{P}_R + T_R^1, \mathbb{P}_R A \mathcal{R}_R = \mathbb{P}_R + T_R^2,$$

where $T_R^j \in \mathcal{K}(L^2(\Gamma)), j = 1, 2.$

Theorem

Let $A \in \mathcal{A}(\Gamma)$. Then A is a Fredholm operator in $L^2(\Gamma)$ if and only if:

(i) A is a locally Fredholm operator;
(ii) All limit operators $A^h \in Lim(A)$ are invertible.

Corollary

Let $A \in \mathcal{A}(\Gamma)$, and A be a locally Fredholm operator. Then

$$sp_{ess} A = \bigcup_{A^h \in Lim(A)} spA^h, \tag{12}$$

where $sp_{ess} A$ is the essential spectrum of A in $L^2(\Gamma)$ that is the set of $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not Fredhom operator in $L^2(\Gamma)$.

The proof of the main theorem on the essential spectrum of quantum graphs is reduced to the this corollary.
We denote by Λ the unbounded operator generated by the Schrödinger operator $-\frac{d^2}{dx^2}$ on $\Gamma\backslash\mathcal{V}$ with domain $\tilde{\mathcal{H}}^2(\Gamma)$. Note that Λ is a nonnegative self-adjoint operator in $L^2(\Gamma)$ and $sp\Lambda \subset [0,\infty)$. Hence the operator $\Lambda_{k^2} = \Lambda + k^2 I : \tilde{H}^2(\Gamma) \to L^2(\Gamma)$ is an isomorphism.
Then we prove that

$$A = \mathcal{H}_q \Lambda_{k^2}^{-1} \in \mathcal{A}(\Gamma), Lim(A) = Lim(\mathcal{H}_q),$$

$$sp_{ess} A = sp_{ess} \mathcal{H}_q,$$

and the theorem on the essential spectrum of the operator $\mathcal{H}_q$ as unbounded in $L^2(\Gamma)$ follows from Corollary 10.